

# Random graph model with power-law distributed triangle subgraphs

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Clustering is well-known to play a prominent role in the description and understanding of complex networks, and a large spectrum of tools and ideas have been introduced to this end. In particular, it has been recognized that the abundance of small subgraphs is important. Here, we study the arrangement of triangles in a model for scale-free random graphs and determine the asymptotic behavior of the clustering coefficient, the average number of triangles, as well as the number of triangles attached to the vertex of maximum degree. We prove that triangles are power-law distributed among vertices and characterized by both vertex and edge coagulation when the degree exponent satisfies  $2 < \beta < 2.5$ ; furthermore, a finite density of triangles appears as  $\beta = 2 + 1/3$ .

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Graph representation is extensively used in many branches of science in order to reduce the complexity of systems whose components have pairwise interactions and where distance is irrelevant. One associates the components of the system with the vertices of a graph and connects two of them by an edge whenever a given property holds. It has turned out that real-world networks, ranging from biology to physics, display common topological features and, importantly, their degrees, power-law distributed (i.e., the number of vertices with  $k$  edges goes as  $k^{-\beta}$  for some  $\beta > 2$ , called the degree exponent), reflect the presence of self-organizing phenomena underlying their architecture [1]. Owing to their power-law degree distribution such networks are usually referred to as scale-free networks [2], i.e., with no intrinsic characteristic degree.

A number of models aiming at understanding the features of complex networks have been proposed, for instance [3, 4, 5, 6, 7] to cite a few. In this work we focus on a model for power-law random graphs [8] giving good insight into the clustering properties. We demonstrate that triangles coagulate into clusters and, in contrast to classical models for random graphs (see [9] for a review), they are power-law distributed: the probability for a randomly selected vertex to participate in  $t$  triangles goes as  $\sim t^{-(1+\beta)/2}$ , with  $\beta$  being the degree exponent. This scaling relation suggests that triangles might be regarded as a fundamental element for the characterization of real-world networks.

Our motivation resides in the recent attention devoted to the occurrence of small subgraphs, or motifs, in scale-free networks. It has been observed [10, 11] that some motifs are over-represented in real-world networks as compared to randomized networks with the same degree distribution. Usually the triangle is the building block of most motifs and for random regular graphs it has been remarked [12] that when one imposes a finite density of triangles, they have the tendency (i.e., higher probability) to organize themselves into complete subgraphs. Surprisingly, this phenomenon is more likely when the imposed density of triangles is small.

Our interest in triangles is also motivated by their interplay with a simple transitivity relation and the fact that the clustering coefficient can be used for breaking graphs up into clusters carrying coherent information. The clustering coefficient for a given vertex  $i$  with degree  $k_i$  is defined as [3]  $C_i = 2t_i/(k_i^2 - k_i)$ ,

$t_i$  being the number of triangles attached to vertex  $i$ . Clusters are obtained by fixing a threshold value and removing all vertices, and edges incident to them, with  $C_i$  falling below it. This scheme was applied to detect interest communities in the World Wide Web [13] which turned out to be strongly affected by the presence of *co-links*. This means that double edges with opposite direction are part of a triangle with high probability, in line with findings in [11], and thus emerge as the basic unit of transitivity. A similar approach has also been employed to organize lexical information into semantic classes in order to differentiate meanings of ambiguous words [14]. Furthermore, related lines of research [15, 16, 17, 18, 19, 20, 21] have stressed the importance and the abundance of cycles (or loops) in scale-free networks.

*The model.* The best known model for random graphs is the Erdős-Rényi model  $\mathcal{G}(n, p)$  in which every graph consists of  $n$  vertices and each pair is connected by an edge with uniform, independent probability  $p$ . The topology of such graphs, however, shows marked deviations from that observed in real-world networks. For instance, if  $p = O(n^{-1})$  the degrees are Poisson distributed, that is, the probability for a randomly selected vertex to have  $k$  edges is given by [9, 22]  $P(k) = (\lambda^k/k!)e^{-\lambda}$ , where  $\lambda$  is the average degree; furthermore, triangles are almost surely (i.e., with probability equal to one in the asymptotic limit) both edge and vertex disjoint.

Here, we investigate a generalization of the Erdős-Rényi model which exhibits a power-law degree distribution. In our analysis we shall follow closely Refs. [8] to which we refer the reader for more details.

So, consider the set of random graphs  $\mathcal{G}(\mathbf{w})$  in which every graph is specified by the average degree sequence  $\mathbf{w} = (w_1, \dots, w_n)$  arranged in decreasing order:  $w_1 \geq w_2 \geq \dots \geq w_n$ . Two vertices  $i$  and  $j$  are connected with probability  $p_{ij} = w_i w_j / \sum_l w_l = \rho w_i w_j$ , where  $1/\rho = \sum_{l=1}^n w_l$ . Importantly, by setting

$$w_i = c (i + i_0)^{-1/(\beta-1)} \quad (1)$$

the number of vertices with degree  $k$  turns out to be proportional to  $k^{-\beta}$ , and as a result the degrees are power-law distributed with degree exponent  $\beta$ . The constants  $c$  and  $i_0$  appearing in Eq. (1) are determined by the average degree  $d$  and

the maximum degree  $m$ . For  $\beta > 2$  one finds [8]

$$c = d \frac{\beta-2}{\beta-1} n^{1/(\beta-1)} \quad \text{and} \quad 1 + i_0 = n \left( \frac{d \beta - 2}{m \beta - 1} \right)^{\beta-1}.$$

Probability normalization requires that  $m^2 \leq \rho^{-1}$ , and so  $m \leq d^{1/2} n^{1/2}$ . In this model the average degree  $d$  is a free parameter and in the following we will assume that  $d > 1$ ; as a consequence, the maximum degree scales with  $n$  as

$$m \sim n^\alpha \quad \text{and} \quad 0 < \alpha \leq \frac{1}{2}. \quad (2)$$

Remark that  $\alpha$  can be chosen independently of  $\beta$ . Yet, another quantity of interest is the second-order average degree  $\tilde{d} = \rho \sum_i w_i^2$ , in terms of which we shall express most of our results. In the asymptotic limit we have [8]:

$$\tilde{d} = \begin{cases} d \frac{(\beta-2)^2}{(\beta-1)(3-\beta)} \left[ \frac{m}{d} \frac{(\beta-1)}{(\beta-2)} \right]^{3-\beta} & \text{if } 2 < \beta < 3 \\ \frac{d}{2} \log \left( \frac{2m}{d} \right) & \text{if } \beta = 3 \\ d \frac{(\beta-2)^2}{(\beta-1)(\beta-3)} & \text{if } \beta > 3 \end{cases} \quad (3)$$

making apparent the existence of three different regimes as a function of the degree exponent.

*Results.* The average number of triangles  $t_i$  attached to vertex  $i$  is  $t_i = \sum_{j>k} \sum_{j \neq i, k \neq i} p_{ij} p_{jk} p_{ki}$ . This sum may be rearranged as

$$t_i = \frac{1}{2} \rho w_i^2 \left[ \rho^2 \sum_{j,k} w_j^2 w_k^2 - \rho^2 \sum_l w_l^4 - 2w_i^2 \rho^2 \sum_l w_l^2 + 2\rho^2 w_i^4 \right]. \quad (4)$$

In all regimes the leading term arises from the first (double) sum in the right-hand side of the above expression. We find that  $t_i/w_i^2 = \rho \tilde{d}^2/2$  is of order  $O(n^{-1} m^{2(3-\beta)})$  if  $2 < \beta < 3$ , of order  $O(n^{-1} (\log n)^2)$  if  $\beta = 3$ , and of order  $O(n^{-1})$  if  $\beta > 3$ . Neglected terms are at most of order  $O(n^{-1} m^{3-\beta})$  if  $2 < \beta < 3$ , at most of order  $O(n^{-1})$  if  $\beta = 3$ , and at most of order  $O(n^{-1} m^{3-\beta})$  if  $\beta > 3$  [23]. It readily follows that in the asymptotic limit the average clustering coefficient of vertex  $i$  reads

$$C_i = \frac{2t_i}{w_i(w_i-1)} = \frac{\rho(\tilde{d}w_i)^2}{w_i(w_i-1)} = \rho(\tilde{d})^2 [1 + O(w_i^{-1})],$$

and for sufficiently large values of  $w_i$  this can be regarded as independent of the degree of the anchor vertex.  $C_i$  can be interpreted as the probability that two neighbors of a vertex of degree  $w_i$  are joined together by an edge. By making use of Eqs. (2) and (3) one finds how the clustering coefficient scales with the number of vertices  $n$  in the asymptotic limit. The average number of triangles attached to the vertex of maximum degree is simply given by  $t_1 = \rho(\tilde{d}m)^2/2$ . The results in the asymptotic limit are summarized in Tab. I.

The average number of triangles  $T$  is obtained by calculating

$$T = \frac{1}{3} \sum_i t_i = \frac{1}{3!} [(\tilde{d})^3 - 3\tilde{d}\rho^2 \sum_l w_l^4 + 2\rho^3 \sum_l w_l^6].$$

TABLE I: Asymptotic behavior of the clustering coefficient,  $C$ , the average number of triangles,  $T$ , and the number of triangles attached to the vertex of maximum degree,  $t_1$ , as a function of the degree exponent  $\beta$ . Recall that  $m \sim n^\alpha$  with  $0 < \alpha \leq 1/2$ .

$2 < \beta < 3$	$\beta = 3$	$\beta > 3$
$\sim m^{2(3-\beta)} n^{-1}$	$\sim (\log m)^2 n^{-1}$	$\sim n^{-1}$
$\sim m^{3(3-\beta)}$	$\sim (\log m)^3$	$< \infty$
$\sim m^{2(4-\beta)} n^{-1}$	$\sim m^2 (\log m)^2 n^{-1}$	$m^2 n^{-1}$

As before, the dominant term arises from the first term in the right-hand side of the above expression,  $\tilde{d}^3/3!$ , of order  $O(m^{3(3-\beta)})$  if  $2 < \beta < 3$ , of order  $O((\log n)^3)$  if  $\beta = 3$ , and of order  $O(1)$  if  $\beta > 3$ . The other ones are at most of order  $O(m^{2(3-\beta)})$  if  $2 < \beta < 3$ , at most of order  $O(\log n)$  if  $\beta = 3$ , and at most of order  $O(m^{3-\beta})$  if  $\beta > 3$  [24]. The asymptotic behavior of  $T$  as a function of  $m$  and  $n$  for the different regimes is also shown in Tab. I.

We next address the question of how triangles are distributed over the graph. Starting from (4) a simple calculation proves that the probability for a randomly selected vertex to participate in  $t$  triangles goes as

$$P(t) \sim t^{-\delta} \quad \text{with} \quad \delta = \frac{1+\beta}{2}, \quad (5)$$

and thus triangles are power-law distributed among vertices.

*Discussion.* Some remarks on Tab. I are in order. We see that irrespective of the choice of  $\alpha$ , Eq. (2), the clustering coefficient remains a decreasing function of  $n$  for  $\beta > 2$ , that is always smaller than 1, and thus it preserves its probabilistic interpretation. The number of triangles always diverges with  $n$  in the range  $2 < \beta \leq 3$ , corresponding to the regime observed in real-world networks (see [1] for examples); if instead  $\beta > 3$  then there are a finite number of triangles, as in the Erdős-Rényi model. From Tab. I we can also see that  $\alpha = 1/2$  seems to be a natural choice, and hence we set  $\alpha$  equal to this value from here on.

Eq. (5) is our main result. This scaling relation tells us that with non-negligible probability some vertices participate in a large number of triangles, which implies that they are not scattered over the whole graph, as in the Erdős-Rényi model, but coagulate around some vertices. Further understanding of such a phenomenon can be gained by studying the inequality  $t_i > w_i/2$ , leading to  $i < O(1) \times n(\tilde{d}^2/n)^{\beta-1} - i_0$ . Triangles start sharing a common edge when  $(n/i_0) \times (\tilde{d}^2/n)^{\beta-1}$  is at least of order  $O(1)$ , that is, for  $2 < \beta < 2.5$ . Furthermore, the number of vertices at which edge coagulation occurs goes as  $n^{-(\beta-\beta_-)(\beta-\beta_+)} + O(n^{(3-\beta)/2})$  with  $\beta_{\pm} = (3 \pm \sqrt{5})/2$ .

Note that as  $\beta$  approaches 2 the vertex of maximum degree sees around itself a tightly connected cloud since the clustering coefficient is close to being constant, whereas for  $\beta \geq 3$  triangles are sparse in its neighborhood. In contrast, by look-

ing at the fraction of triangles attached to it, that is

$$\frac{t_1}{T} \sim \begin{cases} n^{(\beta-3)/2} & \text{if } 2 < \beta < 3 \\ (\log n)^{-1} & \text{if } \beta = 3 \\ < \infty & \text{if } \beta > 3 \end{cases}$$

we deduce that triangles are spread over the graph for  $2 < \beta < 3$ , and essentially centered around the vertex of maximum degree otherwise. Another quantity of interest is the density of triangles, namely  $T/n \sim n^{3(2-\beta)/2+1/2}$  for  $2 < \beta < 3$ , and as  $\beta = 2 + 1/3$  we have a finite density of triangles.

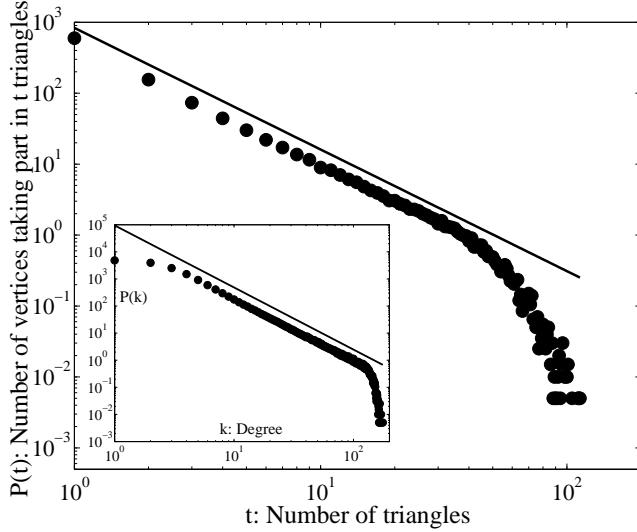


FIG. 1: The number of vertices participating in a given number of triangles as obtained from simulations for  $\beta = 2.2$ . The number of vertices in graphs is set to  $n = 2 \times 10^4$ , the maximum degree to  $m = \sqrt{n}$ , and the average degree to  $d = (\beta - 1)/(\beta - 2)$ . Averages are taken over 200 realizations and the scale of axes is logarithmic. The linear fit yields  $\delta_m = 1.71 \pm 0.02$ . For other values of  $\beta$  the results are summarized in Tab. II. Inset: The degree distribution  $P(k)$ . The solid line has slope  $-2.29 \pm 0.02$ .

Simulations have been performed in order to study the scaling relation of Eq. (5) as a function of  $\beta$ . Figure 1 illustrates the results for  $\beta = 2.2$ . Points obtained from simulations clearly follow a power-law with a cut-off as  $t$  approaches  $t_1 \approx 56$ ; the measured exponent  $\delta_m$  is in accordance with the theoretical value. Table II shows the results for other values of  $\beta$ . Finite size effects are more marked as  $\beta$  approaches 3. The reason is that the number of triangles and, in particular,  $t_1$ , which determines the cut-off, increase with  $n$  at a slower rate (see Tab. I). In that respect it is worth noticing that for  $\beta = 2.2$  and  $n = 2 \times 10^4$  vertices we have  $m \approx 141$  and  $t_1 \approx 56$ , and edge coagulation does not occur since  $t_1 > m/2$  does not hold. This is a finite size effect since for  $n = 10^7$  vertices we would have  $m \approx 3,162$  and  $t_1 \approx 4,033$ , and the condition for edge coagulation is fulfilled. To make this point clearer we have investigated numerically  $t_1$  as a function of  $n$ ; the results are shown in Fig. 2 and we see a good agreement between simulations and theoretical predictions in the different regimes. Ob-

viously, the power-law behavior breaks down in the presence of a small, finite number of triangles on average, i.e.  $\beta > 3$ .

TABLE II: The exponent characterizing the distribution of triangles among vertices, Eq. (5), resulting from simulations as a function of the degree exponent  $\beta$ . Here  $\delta_m$  and  $\delta_t$  denote the measured and theoretical values, respectively.

$\beta$	2.2	2.3	2.5	2.8
$\delta_m$	$1.71 \pm 0.02$	$1.83 \pm 0.04$	$2.05 \pm 0.05$	$2.5 \pm 0.13$
$\delta_t$	1.6	1.65	1.75	1.9

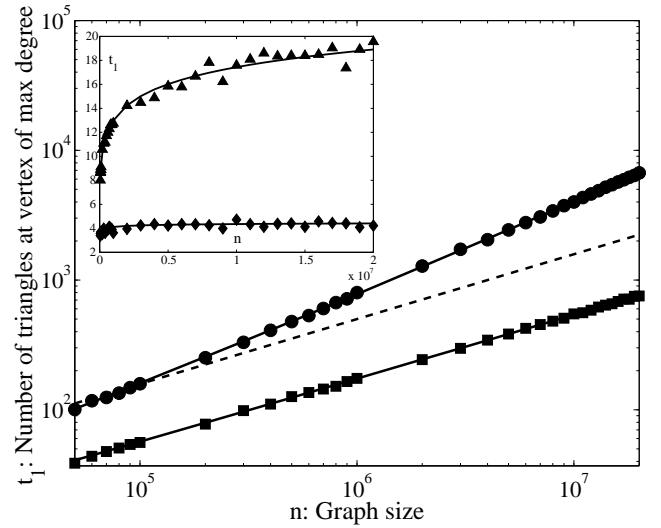


FIG. 2: The dependence of  $t_1$ , the average number of triangles attached to the vertex of maximum degree, on  $n$ , the graph size, as obtained from simulations for  $\beta = 2.2$  (circles) and  $2.5$  (squares). Comparison is made with the theoretical prediction  $t_1 = \rho(\bar{d}m)^2/2$  (solid lines). As before  $d = (\beta - 1)/(\beta - 2)$  and  $m = \sqrt{n}$ ; averages are taken over 200 configurations and the scale of axes is logarithmic. The linear fit yields a slope of  $0.71 \pm 0.01$  for  $\beta = 2.2$  and of  $0.48 \pm 0.01$  for  $\beta = 2.5$ ; the theoretical value is given by  $3 - \beta$  (cf. Tab. I). The dashed line corresponds to  $t_1 = \sqrt{n}/2$  marking the transition to edge coagulation. Notice that for  $\beta = 2.5$  there is no edge coagulation, whereas for  $\beta = 2.2$  there is; indeed, points obtained from simulations cross the dashed line. Inset:  $t_1$  as a function of  $n$  for  $\beta = 3$  (triangles) and  $3.4$  (diamonds). Solid lines correspond to the theoretical predictions.

We point out that the coagulation phenomenon reported in [12] and the one investigated here are of quite a different character. Specifically, in regular graph models the number of graphs with a finite density of triangles is small and correspond statistically to graphs obtained by placing triangles so that to construct the largest complete subgraph. Conversely, in power-law random graphs it turns out that triangles are significant on average and display statistical regularities. The common feature is that as topology departs from a certain degree of randomness it gives rise to a pressure towards clustering and triangles arrange themselves accordingly.

It is possible to make contact with models making use of fitness variables. In Refs. [7, 25] two vertices  $i$  and  $j$  are connected with probability  $f(x_i, x_j)$ , where  $x_i$  and  $x_j$  denote the intrinsic fitness of  $i$  and  $j$ , respectively. Fitness of vertices is distributed according to  $h(x)$ . Within this model the number of triangles attached to a vertex of fitness  $x$  is

$$t(x) = \frac{n^2}{2} \int_0^\infty f(x, y) f(y, z) f(z, x) h(y) h(z) dy dz = \frac{n^2}{2} G(x) .$$

It follows that the probability for a randomly selected vertex to participate in  $t$  triangles can be written as

$$P(t) = h\left[G^{-1}\left(\frac{2t}{n^2}\right)\right] \frac{d}{dt} G^{-1}\left(\frac{2t}{n^2}\right) .$$

The statistical properties of graphs arise from the choice of  $f$  and  $h$  and one can prove that for a particular choice this model is equivalent to the one studied here. We leave a detailed discussion to a future publication.

A generalization of the model investigated here would consist in implementing a non-trivial dependence of the clustering coefficient on the degree. Note, however, that the mechanisms responsible for clustering are basically the same, and in the case of a clustering coefficient decreasing with the degree  $k$  as  $C \sim k^{-\gamma}$  we have  $P(t) \sim t^{-(1+\beta-\gamma)/(2-\gamma)}$ . We address the

reader to Ref. [26] for a study of the presence of this scaling relation in biological networks. The purpose of [26] was to establish a duality between large-scale topological organization and local subgraph structure in empirical networks. Our analysis differs from [26] in that we have dealt with a probabilistic model allowing for a rigorous treatment of the asymptotic limit, but this is done at the expense of generality. Note that random growth processes have been investigated within the framework of the same ideas in [27].

To summarize, in this work we have presented the study of a random graph model and derived the asymptotic behavior of some quantities describing the clustering properties, coming to the conclusion that they are characterized by three regimes, Tab. I. The picture that emerges is that as the degree exponent  $\beta$  decreases the number of triangles increases and arrange themselves into graphs so as to create tightly connected cores around vertices of progressively smaller degree, resulting in a power-law distribution, Eq. (5). This is what we refer to as coagulation of triangles. In itself, this phenomenon dictates the abundance of recurring small patterns in the graph.

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- [24] More precisely, this last estimate holds in the interval  $3 < \beta < 5$ . For  $\alpha = 1/2$ , i.e. the case to which we shall specialize our analysis in the following, we have that neglected terms are of order  $O(n^{-1} \log n)$  if  $\beta = 5$ , and of order  $O(n^{-1})$  if  $\beta > 5$ .
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